Postbuckling of Orthotropic Composite Plates Loaded in Compression

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The nonlinear large-deflection equations of von Kármán are written for "specially" orthotropic plates. The equations are then manipulated to determine the parameters required to establish postbuckling behavior. It is found that only two new parameters are needed beyond those required for buckling. By assuming trigonometric functions in one direction, the plate equations are converted into ordinary nonlinear differential equations that are solved numerically using a two-point boundary problem solver that makes use of Newton's method. The postbuckling behavior is obtained for simply supported and clamped, long, rectangular, orthotropic plates covering the complete range of dimensions and material properties.

Nomenclature

$A_{11},A_{22},A_{12},A_{66}$ = orthotropic plate extensional stiffness
a,b = dimensions of rectangular plate parallel to x and y axes, respectively
b_e = effective width, $b_e=b$, $P\Delta_{cr}/P_{cr}\Delta$
$D_{11},D_{22},D_{12},D_{66}$ = orthotropic plate bending stiffness
F = stress function
$\bar{F}$ = dimensionless stress function, $\bar{F}=F/\sqrt{D_{11}D_{22}}$
k_x,k_y,k_{xy} = curvatures in plate
M_x,M_y,M_{xy} = bending moments in plate
M_{x1},M_{y1},M_{xy1} = functions of y appearing in Eqs. (18) for M_x, M_y, M_{xy}
m = number of half-waves in x direction
m_{cr} = value of m at buckling
N_x,N_y,N_{xy} = in-plane stress resultants in plate
$N_{x0},N_{x2},N_{y0},N_{y2},N_{xy2}$ = functions of y appearing in Eqs. (18) for N_x, N_y, N_{xy}
P = average compressive load per unit length at edge of plate,

$$P=-\frac{1}{b}\int_0^b [N_x]_0^a \mathrm{d}y$$

P_{cr} = value of P at buckling
u,v = displacements in x and y directions, respectively
V_y,β = functions of y defined by Eqs. (23)
w = deflection normal to plate
$\bar{w}$ = nondimensional measure of w, $\bar{w}=w\sqrt{(A_{11}A_{22}-A_{12}^2)/\sqrt{A_{11}A_{22}D_{11}D_{22}}}$
u_2,v_0,v_2,w_1 = arbitrary functions of y appearing in Eqs. (16) for u,v,w
x,y = plate coordinates
$\bar{x},\bar{y}$ = nondimensional plate coordinates, $\bar{x}=x/a$, $\bar{y}=y/b$
Δ = applied end shortening
Δ_{cr} = value of Δ at buckling
$\epsilon_x,\epsilon_y,\gamma_{xy}$ = neutral surface strains in plate

Subscripts x and y following a comma indicate partial differentiation of the principal symbol with respect to the subscripts. Primes indicate differentiation with respect to y.

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Introduction

To take advantage of composites of various ply thicknesses and orientations, it is necessary to establish design methods for a variety of structural configurations and loadings. One of the basic structural configurations is the rectangular flat plate supported at its edges and subjected to an in-plane compressive loading. Nondimensional parameters have been found (e.g., Ref. 1) that allow buckling results to be presented as a series of curves on one plot for each set of edge conditions that cover the complete range of dimensions and material properties.

Although buckling is an important measure of loading, it may be advantageous to determine if a panel can carry considerable load beyond buckling. Postbuckling behavior can be obtained for each configuration using a general-purpose computer program such as that described in Ref. 2. Different materials and ply orientations lead to widely different postbuckling behavior. It would be convenient if postbuckling behavior could also be identified in terms of nondimensional parameters so that each of the loads or deformations of interest could be calculated once to cover the complete range of dimensions and material properties and then presented in a set of curves. The designer could then determine postbuckling behavior of any given panel by interpolation and simple calculations. He could also determine the optimum panel to satisfy his needs by changing dimensions and then determining the strains in the panel associated with each change.

Agreement between theoretical results and experimental results has been obtained for isotropic plates, and similar agreement has been obtained for composites. Many articles have appeared in the literature on the postbuckling of orthotropic composite plates loaded in compression (e.g., Refs. 3-6). None have attempted to set up a set of curves in terms of parameters that would apply to the complete range of dimensions in the postbuckling range.

In the present paper, the large-deflection plate equations of von Kármán (see Ref. 1, p. 347) are written for "specially" orthotropic plates. The equations are then manipulated to determine the parameters required to establish postbuckling behavior. It is found that only two new parameters are needed beyond those required for buckling. The von Kármán equations are nonlinear partial differential equations that are converted into ordinary (nonlinear) differential equations in this paper by assuming trigonometric functions in one direction. These equations are then solved numerically using the method of Ref. 7. In postbuckling analysis, it is necessary to satisfy in-plane boundary conditions in addition to the out-of-plane boundary conditions required for buckling. The in-plane condition satisfied here corresponds to all edges that are straight and free of shear stress. The trigonometric functions assumed were chosen to satisfy the in-plane condition and a simple support condition on the short edges. The solution of the ordinary differential equations derived satisfy the in-plane condition and either a simple support or clamped condition on the long edges. The present paper shows that only one of the two new parameters identified governs the postbuckling behavior of long orthotropic compression-loaded plates. By use of this parameter, plots of load, deflection, end shortening, maximum strains, and effective width are presented covering the complete range of dimensions and material properties.

Parameters Governing Postbuckling Behavior

Dimensionless parameters are determined for postbuckling behavior of orthotropic plates by examining the nonlinear plate equations of von Kármán. By converting each of the independent and dependent variables of the equations into quantities of unit order of magnitude without changing equilibrium, the dimensionless parameters appear as coefficients. To present results in an efficient manner, a minimum number of dimensionless parameters is sought. Equally valid parameters may appear in different forms.

Buckling Parameters

To demonstrate how parameters may be determined, the problem of the buckling of orthotropic plates in compression is used as an example. The buckling equation may be written

$$D_{11} w,_{xxxx} + 2(D_{12}+2D_{66}) w,_{xxyy} + D_{22} w,_{yyyy} + P_{\mathrm{cr}} w,_{xx} = 0 \quad (1)$$

where P_{cr} is the compressive direct stress intensity in the plate just prior to buckling, D_{ij} the various bending stiffnesses of the plate, and w the buckling mode. Other equations of importance are the boundary conditions of the problem, but since they are homogeneous they do not enter into determination of the parameters.

To convert the dependent variables x and y to unit order of magnitude in the interval (a, b), the variables $\bar{x}$ and $\bar{y}$, where

$$\bar{x}=x/a, \qquad \bar{y}=y/b \quad (2)$$

are used, and Eq. (1) becomes

$$\frac{D_{11}}{a^4} w,_{\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{2(D_{12}+2D_{66})}{a^2b^2} w,_{\bar{x}\bar{x}\bar{y}\bar{y}} + \frac{D_{22}}{b^4} w,_{\bar{y}\bar{y}\bar{y}\bar{y}} + \frac{P_{\mathrm{cr}}}{a^2} w,_{\bar{x}\bar{x}} = 0 \quad (3)$$

Multiplication of Eq. (3) by $a^2b^2/\sqrt{D_{11}D_{22}}$ gives

$$\frac{b^2}{a^2}\sqrt{\frac{D_{11}}{D_{22}}} w,_{\bar{x}\bar{x}\bar{x}\bar{x}} + \frac{2(D_{12}+2D_{66})}{\sqrt{D_{11}D_{22}}} w,_{\bar{x}\bar{x}\bar{y}\bar{y}} + \frac{a^2}{b^2}\sqrt{\frac{D_{22}}{D_{11}}} w,_{\bar{y}\bar{y}\bar{y}\bar{y}} + \frac{P_{\mathrm{cr}}b^2}{\sqrt{D_{11}D_{22}}} w,_{\bar{x}\bar{x}} = 0 \quad (4)$$

The buckling results for any orthotropic plate can then be found from Eq. (4) and given in terms of a buckling stress coefficient, often written $P_{\mathrm{cr}}b^2/\pi^2\sqrt{D_{11}D_{22}}$, and as a function of the aspect ratio parameter $(a/b)\sqrt[4]{D_{22}/D_{11}}$, and the twisting stiffness parameter $(D_{12}+2D_{66})/\sqrt{D_{11}D_{22}}$.

Postbuckling Parameters

A convenient way to identify the parameters needed to determine postbuckling behavior beyond those needed for buckling is to examine the plate equilibrium and compatibility equations after they are reduced to two equations in terms of the deflection and stress function. The nonlinear strains considered are

$$\epsilon_x = u,_x + \tfrac{1}{2} w,_x^2$$
$$\epsilon_y = v,_y + \tfrac{1}{2} w,_y^2$$
$$\gamma_{xy} = u,_y + v,_x + w,_x w,_y \quad (5)$$

By differentiation, u and v can be eliminated to give the well-known compatibility equation

$$\epsilon_{x,yy} + \epsilon_{y,xx} - \gamma_{xy,xy} = w,_{xy}^2 - w,_{xx} w,_{yy} \quad (6)$$

The force resultants satisfy equilibrium

$$N_{x,x} + N_{xy,y} = 0, \qquad N_{y,y} + N_{xy,x} = 0 \quad (7)$$

if a stress function F exists so that

$$N_x = F,_{yy}, \qquad N_y = F,_{xx}, \qquad N_{xy} = -F,_{xy} \quad (8)$$

Hooke's law for an orthotropic material relates the force resultants to the strains according to

$$N_x = A_{11}\epsilon_x + A_{12}\epsilon_y, \quad N_y = A_{22}\epsilon_y + A_{12}\epsilon_x, \quad N_{xy} = A_{66}\gamma_{xy} \quad (9)$$

where A_{ij} are the extensional stiffnesses of the plate and, therefore,

$$\epsilon_x = (A_{22}F,_{yy} - A_{12}F,_{xx})/(A_{11}A_{22} - A_{12}^2)$$
$$\epsilon_y = (A_{11}F,_{xx} - A_{12}F,_{yy})/(A_{11}A_{22} - A_{12}^2)$$
$$\gamma_{xy} = -F,_{xy}/A_{66} \quad (10)$$

Entering these expressions in the compatibility equation (6) gives

$$A_{11}F,_{xxxx} + 2\left(\frac{A_{11}A_{22}-A_{12}^2}{2A_{66}} - A_{12}\right)F,_{xxyy} + A_{22}F,_{yyyy} = (A_{11}A_{22}-A_{12}^2)(w,_{xy}^2 - w,_{xx}w,_{yy}) \quad (11)$$

The normal equilibrium equation in terms of the stress function is

$$D_{11} w,_{xxxx} + 2(D_{12}+2D_{66}) w,_{xxyy} + D_{22} w,_{yyyy} = F,_{yy} w,_{xx} + F,_{xx} w,_{yy} - 2F,_{xy} w,_{xy} \quad (12)$$

Equations (11) and (12) together with the boundary conditions determine the postbuckling behavior.

Changing the dependent variables to $\bar{x}$ and $\bar{y}$, as was done in the buckling equation, and multiplying the first equation by $a^2b^2/\sqrt{A_{11}A_{22}}$ and the second by $a^2b^2/\sqrt{D_{11}D_{22}}$ gives

$$\frac{b^2}{a^2}\sqrt{\frac{A_{11}}{A_{22}}}F,_{\bar x\bar x\bar x\bar x}+2\left(\frac{A_{11}A_{22}-A_{12}^2-2A_{12}A_{66}}{2A_{66}\sqrt{A_{11}A_{22}}}\right)F,_{\bar x\bar x\bar y\bar y}$$

$$+\frac{a^2}{b^2}\sqrt{\frac{A_{22}}{A_{11}}}F,_{\bar y\bar y\bar y\bar y}=\frac{A_{11}A_{22}-A_{12}^2}{\sqrt{A_{11}A_{22}}}\left(w,_{\bar x\bar y}^2-w,_{\bar x\bar x}w,_{\bar y\bar y}\right)$$

$$\frac{b^2}{a^2}\sqrt{\frac{D_{11}}{D_{22}}}w,_{\bar x\bar x\bar x\bar x}+2\left(\frac{D_{12}+2D_{66}}{\sqrt{D_{11}D_{22}}}\right)w,_{\bar x\bar x\bar y\bar y}+\frac{a^2}{b^2}\sqrt{\frac{D_{22}}{D_{11}}}w,_{\bar y\bar y\bar y\bar y}$$

$$=\frac{1}{\sqrt{D_{11}D_{22}}}\left(F,_{\bar y\bar y}w,_{\bar x\bar x}+F,_{\bar x\bar x}w,_{\bar y\bar y}-2F,_{\bar x\bar y}w,_{\bar x\bar y}\right) \tag{13}$$

To get some idea of the order of magnitude of the variable F, it can be related back to the buckling stress resultant P_{cr}. That is, at buckling $F=-P_{cr}y^2/2=-P_{cr}\bar y^2b^2/2$. Guided by the buckling stress coefficient, let $\bar F=F/\sqrt{D_{11}D_{22}}$ and, to get a $\bar w$ of order one, let $\bar w=w\sqrt{(A_{11}A_{22}-A_{12}^2)/\sqrt{A_{11}A_{22}D_{11}D_{22}}}$.

Equations (13) then become

$$\frac{b^2}{a^2}\sqrt{\frac{A_{11}}{A_{22}}}\bar F,_{\bar x\bar x\bar x\bar x}+2\left(\frac{A_{11}A_{22}-A_{12}^2-2A_{12}A_{66}}{2A_{66}\sqrt{A_{11}A_{22}}}\right)\bar F,_{\bar x\bar x\bar y\bar y}$$

$$+\frac{a^2}{b^2}\sqrt{\frac{A_{22}}{A_{11}}}\bar F,_{\bar y\bar y\bar y\bar y}=\bar w,_{\bar x\bar y}^2-\bar w,_{\bar x\bar x}\bar w,_{\bar y\bar y}$$

$$\frac{b^2}{a^2}\sqrt{\frac{D_{11}}{D_{22}}}\bar w,_{\bar x\bar x\bar x\bar x}+2\left(\frac{D_{12}+2D_{66}}{\sqrt{D_{11}D_{22}}}\right)\bar w,_{\bar x\bar x\bar y\bar y}$$

$$+\frac{a^2}{b^2}\sqrt{\frac{D_{22}}{D_{11}}}\bar w,_{\bar y\bar y\bar y\bar y}=\bar F,_{\bar y\bar y}\bar w,_{\bar x\bar x}+\bar F,_{\bar x\bar x}\bar w,_{\bar y\bar y}-2\bar F,_{\bar x\bar y}\bar w,_{\bar x\bar y} \tag{14}$$

The equations have been converted into quantities of unit order of magnitude and the required dimensionless parameters appear as coefficients. In addition to the parameters associated with buckling of orthotropic plates $(a/b)\sqrt[4]{D_{22}/D_{11}}$ and $(D_{12}+2D_{66})/\sqrt{D_{11}D_{22}}$, the post-buckling parameters may be taken to be $A_{22}D_{11}/A_{11}D_{22}$ and $(A_{11}A_{22}-P_{12}^2-2A_{12}A_{66})/2A_{66}\sqrt{A_{11}A_{22}}$. Thus, the post-buckling behavior for any problem that satisfies these differential equations can be expressed in terms of these parameters. [$A_{22}D_{11}/A_{11}D_{22}$ is chosen rather than $(a/b)\sqrt[4]{A_{22}/A_{11}}$ so that the new parameters depend only on material properties not on panel aspect ratio. This is particularly convenient when considering the case of a long plate.]

Analysis

The analysis of the postbuckling behavior of a rectangular orthotropic plate in longitudinal compression is considered. Efficient methods exist for solving nonlinear ordinary differential equations with two-point boundary conditions. Plate analysis leads to partial differential equations such as those developed in the previous section. However, if the unknowns can be represented by a few terms of a series of known functions in one variable with arbitrary functions in the other variable as coefficients, the principle of virtual work can be used to derive ordinary differential equations to replace the partial differential equations of plate theory. A study of the results of the analysis of Ref. 8 indicates that good results may be obtained for one pair of opposite edges simply supported by using a few terms of a trigonometric series in one variable with arbitrary functions in the other variable as coefficients in the expressions for displacements and forces.

The plate has a length a and a width b, the origin of the coordinates is taken as one corner, and the x boundary conditions for the pair of opposite edges that are simply supported, free of shear stress, and straight can be written as

Zero deflection	$w(0,y)=w(a,y)=0$
Zero moment	$M_x(0,y)=M_x(a,y)=0$
Zero shear stress	$N_{xy}(0,y)=N_{xy}(a,y)=0$
Applied displacement	$u(0,y)=\Delta/2;\quad u(a,y)=-\Delta/2$

(15)

Guided by Ref. 8, the form of the unknown displacements are

$$u=-\Delta\left(\frac{x}{a}-\frac{1}{2}\right)+u_2(y)\sin\frac{2m\pi x}{a}$$

$$v=v_0(y)+v_2(y)\cos\frac{2m\pi x}{a}$$

$$w=w_1(y)\sin\frac{m\pi x}{a} \tag{16}$$

From the form assumed for the displacements of Eqs. (16) the present analysis allows more freedom in the y direction than the first approximation in Ref. 8 and for a long plate the author expects that the present results should be as good or better than those obtainable by the second approximation of Ref. 8 (which has slightly more freedom in the x direction than the present analysis). A comparison of strains and deformations with results from the general-purpose computer program of Ref. 2 for isotropic plates indicate excellent agreement. The neutral surface strains [Eqs. (5)] as given by von Kármán nonlinear plate theory are

$$\epsilon_x=-\frac{\Delta}{a}+\frac{2m\pi}{a}u_2\cos\frac{2m\pi x}{a}+\frac{1}{4}\left(\frac{m\pi}{a}\right)^2w_1^2\left(1+\cos\frac{2m\pi x}{a}\right)$$

$$\epsilon_y=v_0'+v_2'\cos\frac{2m\pi x}{a}+\frac{1}{4}w_1'^2\left(1-\cos\frac{2m\pi x}{a}\right)$$

$$\gamma_{xy}=\left(u_2'-\frac{2m\pi}{a}v_2+\frac{1}{2}\frac{m\pi}{a}w_1w_1'\right)\sin\frac{2m\pi x}{a} \tag{17}$$

and the curvatures are

$$k_x=-w,_{xx}=\left(\frac{m\pi}{a}\right)^2w_1\sin\frac{m\pi x}{a}$$

$$k_y=-w,_{yy}=-w_1''\sin\frac{m\pi x}{a}$$

$$k_{xy}=-2w,_{xy}=-2\frac{m\pi}{a}w_1'\cos\frac{m\pi x}{a}$$

From the stress-strain law [Eqs. (9)] for an orthotropic plate the form of the stress resultants can be identified as

$$N_x=N_{x0}(y)+N_{x2}(y)\cos\frac{2m\pi x}{a}$$

$$N_y=N_{y0}(y)+N_{y2}(y)\cos\frac{2m\pi x}{a}$$

$$N_{xy}=N_{xy2}(y)\sin\frac{2m\pi x}{a} \tag{18}$$

and the moments as

$$M_x=D_{11}k_x+D_{12}k_y=M_{x1}(y)\sin\frac{m\pi x}{a}$$

$$M_y=D_{22}k_y+D_{12}k_x=M_{y1}(y)\sin\frac{m\pi x}{a}$$

$$M_{xy}=D_{66}k_{xy}=M_{xy1}(y)\cos\frac{m\pi x}{a}$$

where

$$N_{x0}=A_{11}\left[-\frac{\Delta}{a}+\frac{1}{4}\left(\frac{m\pi}{a}\right)^2 w_1^2\right]+A_{12}\left(v_0'+\frac{1}{4}w_1'^2\right)$$

$$N_{x2}=A_{11}\left[\frac{2m\pi}{a}u_2+\frac{1}{4}\left(\frac{m\pi}{a}\right)^2 w_1^2\right]+A_{12}\left(v_2'-\frac{1}{4}w_1'^2\right)$$

$$N_{y0}=A_{22}\left(v_0'+\frac{1}{4}w_1'^2\right)+A_{12}\left[-\frac{\Delta}{a}+\frac{1}{4}\left(\frac{m\pi}{a}\right)^2 w_1^2\right]$$

$$N_{y2}=A_{22}\left(v_2'-\frac{1}{4}w_1'^2\right)+A_{12}\left[\frac{2m\pi}{a}u_2+\frac{1}{4}\left(\frac{m\pi}{a}\right)^2 w_1^2\right]$$

$$N_{xy2}=A_{66}\left(u_2'-\frac{2m\pi}{a}v_2+\frac{1}{2}\frac{m\pi}{a}w_1 w_1'\right)$$

$$M_{x1}=D_{11}\left(\frac{m\pi}{a}\right)^2 w_1-D_{12}w_1''$$

$$M_{y1}=-D_{22}w_1''+D_{12}\left(\frac{m\pi}{a}\right)^2 w_1$$

$$M_{xy1}=2D_{66}\frac{m\pi}{a}w_1' \tag{19}$$

Thus, from Eqs. (16) and (18) it can be seen that the boundary conditions of Eqs. (15) at $x=0$ and $x=a$ are satisfied.

The virtual work of the system is

$$\delta\Pi=\int_0^b\int_0^a (N_x\delta\epsilon_x+N_y\delta\epsilon_y+N_{xy}\delta\gamma_{xy}$$
$$+M_x\delta k_x+M_y\delta k_y+M_{xy}\delta k_{xy})\,\mathrm{d}x\mathrm{d}y \tag{20}$$

Substitution of Eqs. (17) and (18) into Eq. (20) and integrating over x results in

$$\delta\Pi=\int_0^b\left\{N_{x0}\frac{a}{2}\left(\frac{m\pi}{a}\right)^2 w_1\delta w_1+N_{x2}\frac{a}{2}\left[\frac{2m\pi}{a}\delta u_2\right.\right.$$
$$\left.+\frac{1}{2}\left(\frac{m\pi}{a}\right)^2 w_1\delta w_1\right]+N_{y0}a\left(\delta v_0'+\frac{1}{2}w_1'\delta w_1'\right)$$
$$+N_{y2}\frac{a}{2}\left(\delta v_2'-\frac{1}{2}w_1'\delta w_1'\right)+N_{xy2}\frac{a}{2}\left[\delta u_2'-\frac{2m\pi}{a}\delta v_2\right.$$
$$\left.-\frac{1}{2}\frac{m\pi}{a}(w_1\delta w_1'+w_1'\delta w_1)\right]+M_{x1}\frac{a}{2}\left(\frac{m\pi}{a}\right)^2\delta w_1$$
$$\left.-M_{y1}\frac{a}{2}\delta w_1''-2M_{xy1}\frac{a}{2}\frac{m\pi}{a}\delta w_1'\right\}\mathrm{d}y \tag{21}$$

Integration by parts leads to

$$\delta\Pi=\int_0^b\left\{\frac{a}{2}\left(-N_{xy2}'+\frac{2m\pi}{a}N_{x2}\right)\delta u_2-aN_{y0}'\delta v_0\right.$$
$$-\frac{a}{2}\left(N_{y2}'+\frac{2m\pi}{a}N_{xy2}\right)\delta v_2$$
$$+\frac{a}{2}\left[V_y'+\left(\frac{m\pi}{a}\right)^2 M_{x1}+\frac{1}{2}\left(\frac{m\pi}{a}\right)^2(2N_{x0}+N_{x2})w_1\right.$$
$$\left.\left.+\frac{1}{2}\frac{m\pi}{a}N_{xy2}\beta\right]\delta w_1\right\}\mathrm{d}y+\frac{a}{2}\left[N_{xy2}\delta u_2+2N_{y0}\delta v_0\right.$$
$$\left.+N_{y2}\delta v_2-M_{y1}\delta\beta-V_y\delta w_1\right]_0^b \tag{22}$$

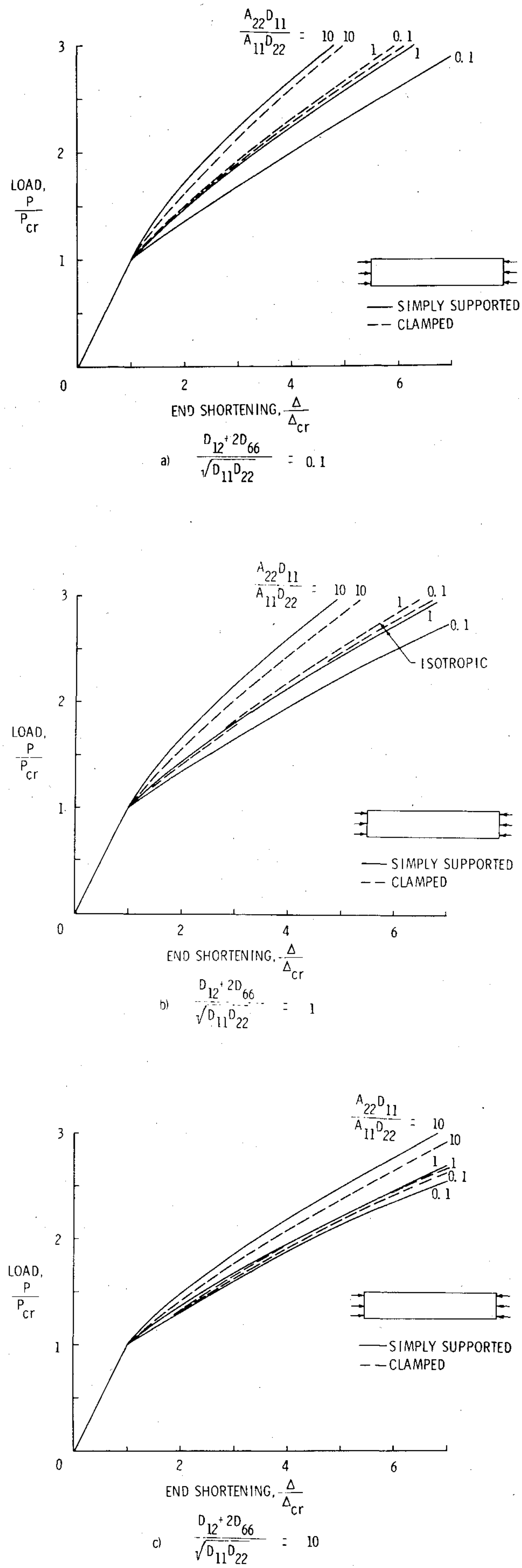

Fig. 1 Normalized load-end-shortening curves for the postbuckling in compression of long, orthotropic plates.

where

$$V_y = -M'_{y1} + 2M_{xy1}\frac{m\pi}{a} - \frac{1}{2}(2N_{y0} - N_{y2})\beta - \frac{1}{2}\frac{m\pi}{a}N_{xy2}w_1$$

$$\beta = w'_1 \tag{23}$$

Thus, the principle of virtual work requires the satisfaction of the following differential equations

$$-N'_{xy2} + \frac{2m\pi}{a}N_{x2} = 0, \quad N'_{y0} = 0, \quad N'_{y2} + \frac{2m\pi}{a}N_{xy2} = 0$$

$$V'_y + \left(\frac{m\pi}{a}\right)^2 M_{x1} + \frac{1}{2}\left(\frac{m\pi}{a}\right)^2 (2N_{x0} + N_{x2})w_1$$

$$+\frac{1}{2}\frac{m\pi}{a}N_{xy2}\beta = 0 \tag{24}$$

and the choice of the following boundary conditions

$$[N_{xy2}u_2]_0^b = 0, \quad [N_{y0}v_0]_0^b = 0, \quad [N_{y2}v_2]_0^b = 0$$

$$[M_{y1}\beta]_0^b = 0, \quad [V_y w_1]_0^b = 0 \tag{25}$$

Since the variations must satisfy the same boundary conditions as the "actual" displacements, the variation signs were omitted in the boundary terms above. For the y boundaries, the simply supported, shear-stress-free, straight condition requires that at $y = 0, b$,

$$v_2 = w_1 = N_{y0} = N_{xy2} = M_{y1} = 0$$

and the clamped, shear-stress-free, straight condition requires that at $y = 0, b$,

$$v_2 = w_1 = \beta = N_{y0} = N_{xy2} = 0$$

Thus, $N_{y0} = 0$ throughout the region. The simultaneous first-order differential equations to be solved can be written in terms of the unknowns u_2, v_2, w_1, β, N_{xy2}, N_{y2}, M_{y1}, and V_y, with v_0 determined by integration after the other unknowns are determined. The following set of differential equations for these unknowns can be determined by simple manipulation of the equations already stated:

$$u'_2 = \frac{N_{xy2}}{A_{66}} + \frac{2m\pi}{a}v_2 - \frac{1}{2}\frac{m\pi}{a}w_1\beta$$

$$v'_2 = \frac{N_{y2}}{A_{22}} + \frac{1}{4}\beta^2 - \frac{A_{12}}{A_{22}}\left[\frac{2m\pi}{a}u_2 + \frac{1}{4}\left(\frac{m\pi}{a}\right)^2 w_1^2\right]$$

$$\beta' = -\frac{M_{y1}}{D_{22}} + \frac{D_{12}}{D_{22}}\left(\frac{m\pi}{a}\right)^2 w_1$$

$$w'_1 = \beta$$

$$N'_{xy2} = \frac{2m\pi}{a}\left\{\frac{A_{12}}{A_{22}}N_{y2} + \left(A_{11} - \frac{A_{12}^2}{A_{22}}\right)\right.$$

$$\left.\times\left[\frac{2m\pi}{a}u_2 + \frac{1}{4}\left(\frac{m\pi}{a}\right)^2 w_1^2\right]\right\}$$

$$N'_{y2} = -\frac{2m\pi}{a}N_{xy2}$$

$$M'_{y1} = -V_y - 4D_{66}\left(\frac{m\pi}{a}\right)^2\beta + \frac{1}{2}N_{y2}\beta - \frac{1}{2}\frac{m\pi}{a}N_{xy2}w_1$$

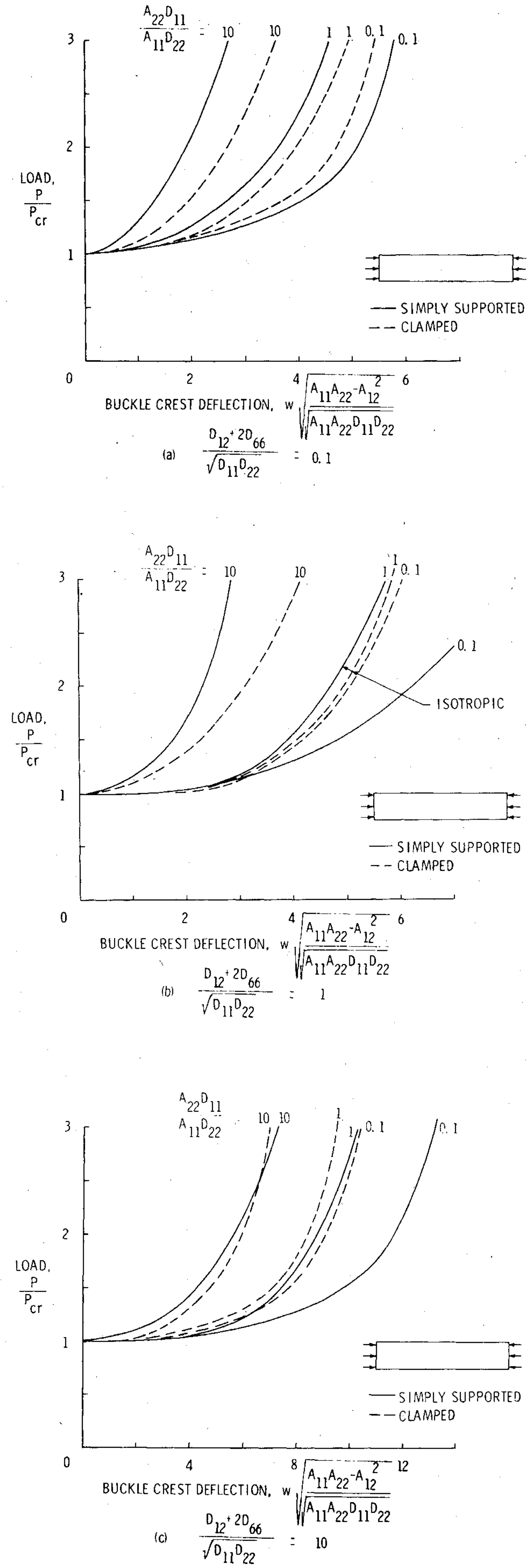

Fig. 2 Load deflection curves for the postbuckling in compression of long, orthotropic plates.

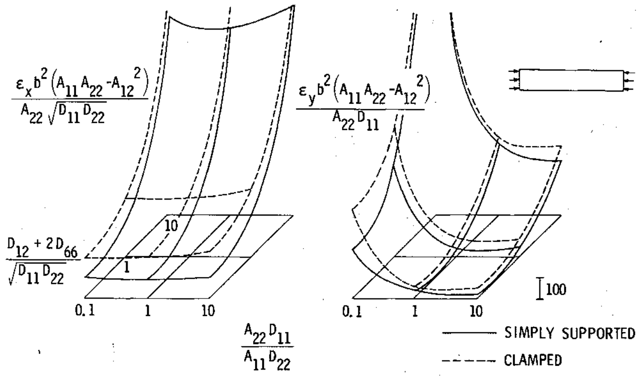

Fig. 3 Maximum values of direct strains for the postbuckling in compression of long, orthotropic plates at $\Delta/\Delta_{cr}=4$.

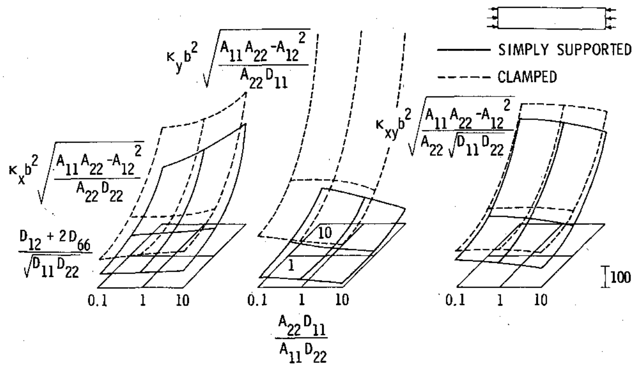

Fig. 4 Maximum values of bending curvatures and twist for the postbuckling in compression of long, orthotropic plates at $\Delta/\Delta_{cr}=4$.

$$V_y' = -\left(\frac{m\pi}{a}\right)^2\left[\frac{D_{12}}{D_{22}}M_{y1} + \left(D_{11}-\frac{D_{12}^2}{D_{22}}\right)\left(\frac{m\pi}{a}\right)^2 w_1\right]$$
$$-\frac{1}{2}\frac{m\pi}{a}N_{xy2}\beta - \frac{1}{2}\left(\frac{m\pi}{a}\right)^2\left\{\frac{A_{12}}{A_{22}}N_{y2} + \left(A_{11}-\frac{A_{12}^2}{A_{22}}\right)\right.$$
$$\left.\times\left[-2\frac{\Delta}{a}+\frac{2m\pi}{a}u_2+\frac{3}{4}\left(\frac{m\pi}{a}\right)^2 w_1^2\right]\right\}w_1 \qquad (26)$$

Equations (26) and both the simply supported and clamped sets of boundary conditions can be solved directly using the algorithm of Ref. 7 and the solution procedure described in the next section.

Solution Procedure

A solution procedure is described for obtaining results in the postbuckling range for long rectangular plates in longitudinal compression. As with many nonlinear problems, it is often best to work from what is known to what is desired. Generally, buckling results are known. The buckling results used in this study are the applied end shortening at buckling Δ_{cr} and m_{cr} the number of half-waves at buckling for a finite, but long, plate. For simply supported, long, orthotropic plates the buckling load is

$$\frac{P_{cr}b^2}{\pi^2\sqrt{D_{11}D_{22}}} = 2+2\left(\frac{D_{12}+2D_{66}}{\sqrt{D_{11}D_{22}}}\right) \qquad (27)$$

and

$$m_{cr} = \frac{a}{b}\sqrt[4]{\frac{D_{22}}{D_{11}}}$$

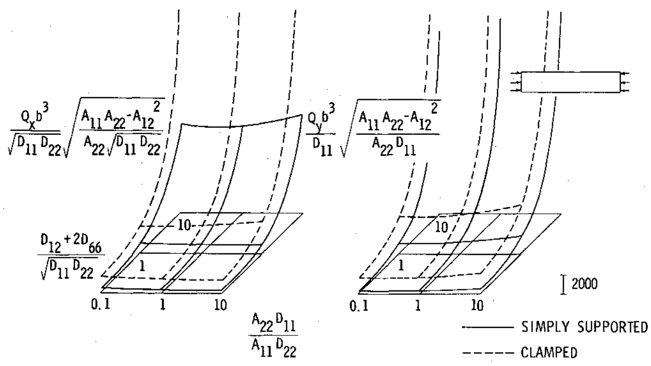

Fig. 5 Maximum values of transverse shearing forces for the postbuckling in compression of long, orthotropic plates at $\Delta/\Delta_{cr}=4$.

For clamped, long, orthotropic plates the buckling load is

$$\frac{P_{cr}b^2}{\pi^2\sqrt{D_{11}D_{22}}} = 4.58+2.4\left(\frac{D_{12}+2D_{66}}{\sqrt{D_{11}D_{22}}}\right) \qquad (28)$$

and

$$m_{cr} = 1.54\frac{a}{b}\sqrt[4]{\frac{D_{22}}{D_{11}}}$$

Note that $\Delta_{cr}/a = P_{cr}A_{22}/(A_{11}A_{22}-A_{12}^2)$ where P_{cr} is given by either Eq. (27) or (28) (obtained from Ref. 9). The applied end shortening for the postbuckling solution will, of course, be larger than the buckling value; also, the wave number will be either equal to or larger than the buckling value and the deflection w will not be zero everywhere. Thus, for a long plate, changes in buckle pattern are accounted for by considering various half-wave numbers until a minimum average load per unit length P is obtained corresponding to the desired applied end shortening, where

$$P = -\frac{1}{b}\int_0^b [N_x]_0^a \mathrm{d}y \qquad (29)$$

Only results for the half-wave number corresponding to the minimum average load are of interest for a long plate with this applied end shortening.

Results and Discussions

Normalized load end shortening results are presented in Fig. 1 for the postbuckling of long rectangular orthotropic plates loaded in longitudinal compression. Results are presented for cases with the short edges shear-stress-free, straight, and simply supported and with the long edges, shear-stress-free, straight, and either simply supported or clamped. The load P resulting from the applied end shortening Δ is normalized by the buckling load P_{cr} and the end shortening is normalized by Δ_{cr}. Buckling of such orthotropic plates is known to depend on the parameters

$$\frac{a}{b}\sqrt[4]{\frac{D_{22}}{D_{11}}} \quad \text{and} \quad \frac{D_{12}+2D_{66}}{\sqrt{D_{11}D_{22}}}$$

The present analysis identifies that the postbuckling results for long plates in longitudinal compression depend on only one new parameter $A_{22}D_{11}/A_{11}D_{22}$. The value of the second new parameter does not change the results for this example. The second new parameter is associated with the in-plane shear stresses that are essentially zero for this problem. Since a long plate is considered, $(a/b)\sqrt[4]{D_{22}/D_{11}} \gg 3$, and continuous changes in the buckling mode are permitted in the analysis (see Refs. 8 and 10). The results presented in Fig. 1 are general in that they represent low, medium, and high values of the parameters $(D_{12}+2D_{66})/\sqrt{D_{11}D_{22}}$ and $A_{22}D_{11}/$

$A_{11}D_{22}$ and by interpolation they apply to essentially the complete range of dimensions for any "specially" orthotropic plate including many practical stacking sequences for laminated composites.

The slope of the load-end-shortening curve is a measure of the overall plate stiffness. As shown in Fig. 1, this curve is a straight line of slope equal to one prior to buckling. After buckling this line may change slope, depending on the values of the parameter chosen. For high values of the $A_{22}D_{11}/A_{11}D_{22}$ parameter, the initial postbuckling slope for simply supported edges does not change from one. For values of this parameter near one, the initial postbuckling slope for simply supported edges is one-half. For low values, the initial postbuckling slope for simply supported edges is one-third. Similar initial postbuckling slopes occur for clamped edges. The slope of the curves decreases with increase in loading. The curves representing the isotropic case appear on Fig. 1b where the values of both $(D_{12}+2D_{66})/\sqrt{D_{11}D_{22}}$ and $A_{22}D_{11}/A_{11}D_{22}$ are unity.

The effective width b_e may also be of interest for design purposes. It can be identified as

$$b_e/b = P\Delta_{cr}/P_{cr}\Delta$$

and, therefore, can be determined directly once the load P/P_{cr} for a given end shortening Δ/Δ_{cr} has been determined from Fig. 1.

Load P normalized by the buckling load P_{cr} is plotted as a function of the normalized out-of-plane deflection w at the crest of the buckle in Fig. 2 for the same range of parameters as used for end shortening in Fig. 1. In every case, the deflection starts from zero at the buckling load with zero slope. After buckling there is a wide variation in the magnitude of the deflection for different values of the parameters and for the load range shown. (Note the change in the deflection scale for Fig. 2c.) Judging from the trend of these curves for higher loads, less variation in deflection is expected for each curve. For the isotropic curves, shown on Fig. 2b, the parameter for the buckle crest deflection can be identified as $2\sqrt{3(1-\mu^2)}\,w/t$, where μ is Poisson's ratio of the isotropic material and t is the plate thickness. Based on results not shown here the shape of the deflection curves is similar to that predicted by Koiter.[11]

Typical maximum values of nondimensional strains, curvatures, and shearing forces obtained from the analysis for various values of the parameters $A_{22}D_{11}/A_{11}D_{22}$ and $(D_{12}+2D_{66})/\sqrt{D_{11}D_{22}}$ are plotted in Figs. 3-5, respectively, for the loading where the applied end shortening is four times the end shortening for buckling, i.e., $\Delta/\Delta_{cr}=4$. The maximum values for the strains ϵ_x and ϵ_y occur in the panel corners (compressive for ϵ_x and tensile for ϵ_y). No curves are given for the shearing strain γ_{xy} since the shearing strain is essentially zero everywhere. The maximum values for the longitudinal curvature k_x occurs at the crest of a buckle. The maximum values for the other direct curvature k_y also occurs at midbuckle length, but at the quarter points for simply supported edges and at the edge for clamped edges. The maximum value of twist k_{xy} occurs at the panel edge at the node of the buckle for simply supported edges and at the quarter points along the node for clamped edges. The maximum value of the shear force Q_x is at the quarter points along the node for simply supported edges and at the panel corner for clamped edges. The maximum value of the shear force Q_y occurs at the panel edge at midbuckle length for both boundary conditions.

Conclusions

The large deflection equations of von Kármán for orthotropic plates are examined to determine the parameters required to establish postbuckling behavior. It is determined that only two new parameters are needed beyond those required for buckling and, for long plates in longitudinal compression, only one new parameter is needed. Results are presented in normalized form for long composite plates to represent the complete range of dimensions and material properties for both simply supported and clamped edges.

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